1. **Problem:** Let X be a normed linear space. Show that X^* is a Banach space.
Solution: See the proposition 2.3.3 from the book "Functional Analysis" by S. Kesavan. It tells that for any two normed linear spaces V, W the set of all bounded linear map from V to W is a Banach space if W is a banach space. For dual space of a normed linear space the codomain is $\mathbb{C}$ which is a Hilbert space and hence we are done.

2. **Problem:** State the Open mapping theorem. State and prove the Closed graph theorem.
Solution: Let X, Y are two Banach spaces and $A \in \mathcal{B}(X,Y)$. If A is surjective, then it is open i.e. it maps open set to open set.
For Closed graph theorem see page number: 44 from the book "Note on Functional Analysis" by Rajendra Bhatia.

3. **Problem:** Let $\{f_n\}_{n\geq 1} \subset L^4([0,1])$ be such that $\|f_n\| \to 0$. Show that for any $g \in L^{\frac{4}{3}}([0,1])$, $\int f_n g dx \to 0$.
Solution: $|\int f_n g dx| \leq \int |f_n g| dx \leq \|f_n\|_4 \|g\|_{\frac{4}{3}}$ using Holder's inequality.
But, it is given that $\|g\|_{\frac{4}{3}}$ is bounded so $\|f_n\|_4\|g\|_{\frac{4}{3}} \to 0$ as $n \to \infty$ i.e. $\int f_n g dx \to 0$ as $n \to \infty$.

4. **Problem:** Let A be a commutative Banach algebra with identity e. Let I be a proper closed ideal. Show that the qutient space A/I is a banach algebra.
Solution: As, I is a proper ideal A/I is nonzero. From page number 20 of the book "Note on Functional Analysis" by Rajendra Bhatia, we know that A/I is a Banach space with respect to the norm defined by $\|x+I\| := inf\{\|x+i\| : i \in I\}$ where $x \in A$.
Multiplication in A/I is given by $(x_1+I)(x_2+I) := (x_1x_1+I)$. As A is a Banach algebra, multiplication in A/I is associative. So, only remaining part is $\|(x_1+I)(x_2+I)\| \leq \|(x_1+I)\|\|(x_2+I)\|$ i.e. to show that $\|(x_1 5 x_2+I)\| \leq \|(x_1+I)\|\|(x_2+I)\|$.
For $i_1, i_2 \in I$, $(x_1+i_1)(x_2+i_2) = (x_1x_2 + x_1i_2 + i_1x_2 + i_1i_2) \in (x_1x_2+I)$.
$\|x_1x_2+I\| := inf\{\|x_1x_2+i\| : i \in I\} \leq inf\{\|(x_1+i_1)(x_2+i_2)\| : i_1, i_2 \in I\} \leq inf\{\|(x_1+i_1)\|\|(x_2+i_2)\| : i_1, i_2 \in I\} = \|(x_1+I)\|\|(x_2+I)\|$.

5. **Problem:** Show that any finite dimensional subspace of a normed linear space is closed.
Solution: See Corollary 2.3.2 from the book "Functional Analysis" by S. Kesavan.

6. **Problem:** State and prove the Banach-Alaoglu theorem.
Solution: See page no. 74 of the book "Note on Functional Analysis" by Rajendra Bhatia.

7. **Problem:** Show that any separable Hilbert space is isomorphic to l^2.
Solution: See Theorem 10. from page no. 96 of the book "Note on Functional Analysis" by Rajendra Bhatia.

8. **Problem:** Let A be a Banach algebra with identity e. Show that for any complex homomorphism $\phi : A \to \mathbb{C}$, kerϕ is a closed ideal.
Solution: We first show that $I :=$ kerϕ is an ideal of A.
Let $x \in$ kerϕ and $a \in A$, then $\phi(ax) = \phi(a)\phi(x) = 0$ and also $\phi(xa) = \phi(x)\phi(a) = 0$ i.e. $ax, xa \in$kerϕ and hence I is a two sided ideal of A.
Now, we will show that ϕ is continous. As, for $a \in A$, $\phi(a - \phi(a)) = 0$ therefore $\phi(a) \in$ sp(a) i.e. $|\phi(a)| \leq \|a\|$, which shows that ϕ is continuous.

To show that, I is closed, consider a sequence $\{x_n\} \in I$ coverging to $x \in A$. Now, as ϕ is continuos we get, $\phi(x) = \lim(\phi(x_n)) = 0$ i.e. $x \in I$. Hence, we are done.

9. **Problem:** Show that any unitary operator on a complex Hilbert space is an isometry and preserves the the inner product.
Solution: $U \in \mathcal{B}(H)$ is said to be unitary if $U^*U = I$ and $UU^* = I$. First condition tells that U is an isometry and using Polarisation identity we show that $U^*U = I$ iff $< Ux, Uy >=< x, y >$ for any $x, y \in H$.

10. **Problem:** Let $\triangle = \{z : |z| \leq 1\}$. Let $A = \{f \in C(\triangle) : f$ is analytic in the interior$\}$. Show that A is a Banach algebra with identity.
Solution: Multiplication in A is pointwise multiplication and addition is pointwise addition. For $f \in A$ we define norm as $\|f\| := \underset{z\in\triangle}{sup}|f(z)|$.
To show that A is closed in this norm, let a sequnce $\{f_n\}$ converging to f in this sup norm.
It is clear that $f \in \mathbb{C}(\triangle)$ because the convergence is uniform.
But to check that f is analytic in the interior of $\triangle$ we use Moreara's theorem. Let C is a closed curve in $\triangle$.
Now, using uniform convergence of $\{f_n\}$ and holomorphicity of f_n we get
$\oint\limits_C f(z)dz = \oint\limits_C lim f_n(z)dz = lim \oint\limits_C f_n(z)dz = 0.$
Hence, A is closed with respect to the above norm.
Let $f, g \in A$. Then, $\|fg\| = \underset{z\in\triangle}{sup}|fg(z)| = \underset{z\in\triangle}{sup}|f(z)||g(z)| \leq \left(\underset{z\in\triangle}{sup}|f(z)|\right)\left(\underset{z\in\triangle}{sup}|g(z)|\right) = \|f\|\|g\|$.
Therefore, A is a Banach algebra. We call this as disc algebra.